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The large-*N* limit of PT-symmetric $O(N)$ models

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Abstract
We study a PT-symmetric quantum-mechanical model with an $O(N)$-symmetric potential of the form $m^2\vec{x}^2/2 - g(\vec{x}^2)^2/N$ using its equivalent Hermitian form. Although the corresponding classical model has finite-energy trajectories that escape to infinity, the spectrum of the quantum theory is proven to consist only of bound states for all *N*. We show that the model has two distinct phases in the large-*N* limit, with different scaling behaviors as *N* goes to infinity. The two phases are separated by a first-order phase transition at a critical value of the dimensionless parameter $m^2/g^{2/3}$, given by $3 \times 2^{1/3}$.

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1. Introduction

Models with PT symmetry have emerged as an interesting extension of conventional quantum mechanics. There is a large class of models that are not Hermitian, but nevertheless have real spectra as a consequence of PT symmetry. Bender and Boettcher have shown that single-component quantum-mechanical models with PT-symmetric potentials of the form $-\lambda(-\mathrm{i}x)^p$ have real spectra [1]. An extensive literature on PT symmetry and related matters now exists, and there are extensive review articles available [2, 3]. Most of the results have been for models without continuous internal symmetries, but there have been some results on models with $O(N)$ symmetry [4–6]. These models are particularly interesting in the large-*N* limit. Meisinger and Ogilvie have shown that a PT-symmetric version of the $O(N)$-invariant anharmonic oscillator is isospectral with a Hermitian model with an $O(N-1)$ symmetry [5]. We study here the properties of this model, using its Hermitian form.

The Euclidean Lagrangian of the PT-symmetric model with $O(N)$ symmetry is given by

$$L_E = \sum_{j=1}^{N}\left[\frac{1}{2}(\partial_t x_j)^2 + \frac{1}{2}m^2 x_j^2\right] - \frac{g}{N}\left(\sum_{j=1}^{N} x_j^2\right)^2, \tag{1}$$

where *g* is positive. The minus sign in front of *g* would lead to a Hamiltonian unbounded from below if the model were Hermitian. From the standpoint of PT symmetry, the interaction term can be considered as a member of a family of PT-invariant interactions

$$-\frac{g}{N}\left(-\sum_{j=1}^{N} x_j^2\right)^p \tag{2}$$

which are invariant under PT symmetry [1]. This class of models is well defined for $p = 1$, and must be defined for $p > 1$ by an appropriate analytic continuation of the x_j as necessary. In [5], it was shown that this PT-symmetric model is equivalent to a Hermitian model with Euclidean Lagrangian given by

$$L_E = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\vec{\pi}}^2 - m^2\sigma^2 + \frac{4g}{N}\sigma^4 + \frac{16g}{N}\sigma^2\vec{\pi}^2 - \sqrt{2gN}\sigma, \tag{3}$$

where σ is a single variable and $\vec{\pi}$ is a vector of $N - 1$ variables. The corresponding Hamiltonian H is

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2}\frac{\partial^2}{\partial\vec{\pi}^2} - m^2\sigma^2 + \frac{4g}{N}\sigma^4 + \frac{16g}{N}\sigma^2\vec{\pi}^2 - \sqrt{2gN}\,\sigma. \tag{4}$$

This Hermitian form has many remarkable features. First of all, it has a manifest $O(N-1)$ symmetry associated with rotations of $\vec{\pi}$ rather than the $O(N)$ symmetry of equation (1). As in the similar case of a PT-symmetric $-gx^4$ theory of a single variable [7, 8], there is a linear anomaly term which breaks the classical symmetry $\sigma \to -\sigma$ at order $\hbar$. The $\vec{\pi}$ field has no quadratic mass term, while the sign of the mass term for the σ field is opposite the sign of the x fields in the original Lagrangian. From a naive field-theoretic point of view, the $\vec{\pi}$ field is massless at tree level.

There are two important questions we will address. The first question is the presence or absence of scattering states in this model for any value of N. It is clear that the corresponding classical model has a class of trajectories with finite, continuously varying energies that escape to infinity along paths with $\sigma = 0$. Nevertheless, we will prove that the quantum-mechanical model has only bound states with discrete energy levels. This behavior is similar to that of the quantum-mechanical x^2y^2 model. This model can be derived from the quantum-mechanical reduction of a two-dimensional gauge theory [9], and was originally thought to be fully ergodic, i.e., to have only chaotic motion. This turns out not to be the case [10]. Simon has shown that the quantum-mechanical version of the x^2y^2 model has a purely discrete spectrum [11]. As we will show in section 2, the arguments of Simon can be generalized to show that the PT-symmetric $O(N)$ models have discrete spectrum for all finite values of N.

The second, more difficult question, concerns the existence and interpretation of the large-N limit. We explain the nature of the problem in section 3. In section 4, we show that a simple variational approximation gives us the clues we need to prove that this model has two distinct scaling behaviors in the large-N limit, controlled by the dimensionless parameter $m^2/g^{2/3}$. A first-order transition occurs in the large-N limit when $m^2/g^{2/3} = 3 \times 2^{1/3}$. Because this is a quantum-mechanical system, the phase transition only appears in the strict limit of $N \to \infty$. For large but finite N, there is a rapid crossover between the two different scaling behaviors. Section 5 rederives the results of section 4 using the Born–Oppenheimer approximation, which proves to be exact in the large-N limit. In section 6 we present the results of a numerical study of the ground-state energy for $m^2 = 0$ that confirm our analytical results. A final section presents our conclusions.

2. Absence of scattering states

We now turn to the issue of the spectrum for finite N. In [11], Simon gave five arguments for the absence of scattering states in an x^2y^2 potential model. His first, and simplest, argument is based on the Golden–Thompson inequality [12–14], which states that the quantum partition function associated with a Hamiltonian H is bounded by the corresponding classical partition function:

$$Z_q = \mathrm{Tr}\, \mathrm{e}^{-\beta H(p,q)} \leqslant Z_c = \int \frac{\mathrm{d}^N p \mathrm{d}^N q}{(2\pi)^N} \mathrm{e}^{-\beta H(p,q)}. \tag{5}$$

Simon observes that a continuous spectrum associated with H is incompatible with a finite value for Z_q. For example, for plane waves in a box of volume V, both partition functions diverge as V goes to infinity. For the Hamiltonian

$$H_{xy} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x^2 y^2 \tag{6}$$

we have the operator inequality

$$-\frac{\partial^2}{\partial x^2} + x^2 y^2 \geqslant |y| \tag{7}$$

which obviously holds on a harmonic oscillator basis. Applying this inequality symmetrically to x and y, we see that

$$H_{xy} \geqslant \frac{1}{2}\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + |x| + |y|\right]. \tag{8}$$

This inequality tells us that the quantum partition function of the x^2y^2 model is bounded by the quantum partition function of a Hamiltonian for which Z_c is finite. The combination of the operator bound with the Golden–Thompson inequality thus leads to the conclusion that the x^2y^2 model has a discrete spectrum.

Applying the same ideas to our Hamiltonian, we see that

$$H \geqslant -\frac{1}{4}\frac{\partial^2}{\partial \sigma^2} - \frac{1}{4}\frac{\partial^2}{\partial \vec{\pi}^2} - m^2\sigma^2 + \frac{4g}{N}\sigma^4 - \sqrt{2gN}\sigma + \sqrt{\frac{2g}{N(N-1)}}\sum_{j=1}^{N-1}|\pi_j| + (N-1)\sqrt{\frac{2g}{N}}|\sigma|. \tag{9}$$

The specific bounding Hamiltonian is not so important; the crucial feature is that the classical partition function Z_c associated with the bounding Hamiltonian is finite. Then the Golden–Thompson inequality ensures that the quantum partition function of the bounding Hamiltonian is finite as well. This in turn allows us to conclude that Z_q for our Hamiltonian is also finite, and thus has only discrete energy eigenstates for any finite value of N.

3. Scaling arguments and the large-N limit

We now consider the large-N limit of the model. Our naive expectation based on the Hermitian $O(N)$ model is that the ground-state energy will be proportional to N as $N \to \infty$ [15]. In order to explore this possibility, we rescale the Lagrangian L_E by $\sigma \to \sqrt{N}\sigma$ to obtain

$$L_E = \frac{N}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\vec{\pi}}^2 - Nm^2\sigma^2 + 4gN\sigma^4 + 16g\sigma^2\vec{\pi}^2 - N\sqrt{2g}\sigma. \tag{10}$$

We see that the anomaly term survives in the large-N limit, unlike the PT-symmetric matrix case [5]. After integrating over the $N-1$ $\vec{\pi}$ fields, we have a large-N effective potential V_{eff} for σ:

$$V_{\mathrm{eff}}/N = -m^2\sigma^2 + 4g\sigma^4 + \tfrac{1}{2}\sqrt{32g\sigma^2} - \sqrt{2g}\sigma. \tag{11}$$

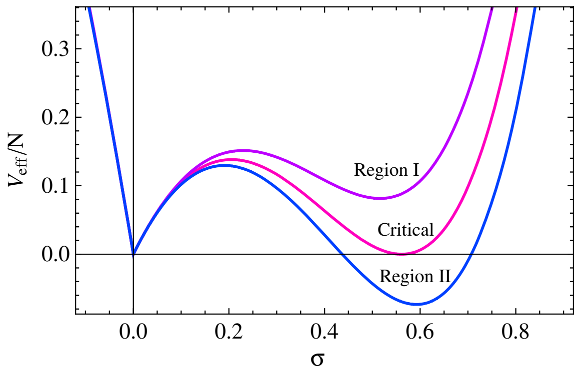

Figure 1. The effective potential $V_{\rm eff}/N$ versus σ for $g = 1$. Three different values of m^2 are shown, corresponding to region I, region II and the critical point.

The new term $\frac{1}{2}\sqrt{32g\sigma^2}$ comes from the functional determinant for fluctuations of the $\vec{\pi}$ fields, and represents the zero-point energy of their quantum fluctuations given a constant value for σ. The shape of the potential is controlled by the dimensionless parameter $m^2/g^{2/3}$. Figure 1 shows the effective potential as a function of σ with g set to 1 for three different values of m^2. We refer to the region where $m^2/g^{2/3} < 3 \times 2^{1/3}$ as region I, and the region where $m^2/g^{2/3} > 3 \times 2^{1/3}$ as region II. In region I, $V_{\rm eff}$ has a global minimum at $\sigma = 0$. For σ near 0, $V_{\rm eff}$ is approximately linear, but with different slopes for $\sigma > 0$ and $\sigma < 0$. This occurs because the zero-point energy of the $\vec{\pi}$ fields has virtually the same form as the anomaly term. However, the zero-point energy term respects a discrete, classical $\sigma \to -\sigma$ symmetry which the anomaly explicitly breaks. The boundary between regions I and II is given by $m^2/g^{2/3} = 3 \times 2^{1/3}$, where $V_{\rm eff}$ has two degenerate minima. In region II, $V_{\rm eff}$ has a global minimum with $\sigma \neq 0$. This change in the behavior of the effective potential as m^2 is varied is not seen in the corresponding Hermitian model [15], and naively indicates that the $N - 1$ $\vec{\pi}$ fields are massless modes in region I.

To understand better the suspect character of the above analysis, it is useful to rederive these results using hyperspherical coordinates [16, 17]. This formalism will also be used later in the context of the Born–Oppenheimer approximation. If we define $\rho = (\vec{\pi}^2)^{1/2}$, the reduced, radial Hamiltonian in the sector of angular momentum l is

$$H_{\rm rad} = -\frac{1}{2}\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2}\frac{\partial^2}{\partial\rho^2} + \frac{(N+2l-3)(N+2l-1)}{8\rho^2} - m^2\sigma^2 + \frac{4g}{N}\sigma^4 + \frac{16g}{N}\sigma^2\rho^2 - \sqrt{2gN}\sigma, \tag{12}$$

where l is a non-negative integer. After rescaling $\sigma \to \sqrt{N}\sigma$ and $\rho \to \sqrt{N}\rho$, we have

$$H_{\rm rad} = -\frac{1}{2N}\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2N}\frac{\partial^2}{\partial\rho^2} + \frac{(N+2l-3)(N+2l-1)}{8N\rho^2} - Nm^2\sigma^2 + 4gN\sigma^4 + 16gN\sigma^2\rho^2 - N\sqrt{2g}\sigma. \tag{13}$$

Taking $l = 0$, we see that the ground-state energy to leading order in large-N is given by minimizing

$$N\left[\frac{1}{8\rho^2} - m^2\sigma^2 + 4g\sigma^4 + 16g\sigma^2\rho^2 - \sqrt{2g}\sigma\right] \tag{14}$$

with respect to ρ and σ. Minimizing with respect to ρ, we find

$$\rho = (128g\sigma^2)^{-1/4} \tag{15}$$

and must now minimize

$$N\,[-m^2\sigma^2 + 4g\sigma^4 + 2\sqrt{2g\sigma^2} - \sqrt{2g}\sigma] \tag{16}$$

which is identical to our previous expression for V_{eff}. However, we now note that when $\sigma = 0$, ρ is infinite. This strongly suggests that our treatment of the large-N limit is not valid in region I.

4. A simple variational approximation

In order to gain analytical insight about the large-N behavior of this model, we apply a simple variational approximation using harmonic oscillator ground states. We assume that the ground state has the form

$$\Psi\,[\sigma, \vec{\pi}] = \psi_0(\sigma) \prod_{j=1}^{N-1} \phi_0(\pi_j), \tag{17}$$

where ϕ_0 is a harmonic oscillator ground state of frequency Ω and expected value $\langle \pi_j \rangle = 0$. The wavefunction $\psi_0(\sigma)$ has frequency ω and expected value $\langle \sigma \rangle = v$. We then have the variational inequality $E_0 \leqslant E_{\text{var}}$ where

$$\begin{aligned} E_{\text{var}} = \frac{1}{4}\omega + \frac{N-1}{4}\Omega - m^2\left(v^2 + \frac{1}{2\omega}\right) + \frac{4g}{N}\left(v^4 + \frac{3v^2}{\omega} + \frac{3}{4\omega^2}\right) \\ + \frac{16g}{N}\left(v^2 + \frac{1}{2\omega}\right)\frac{N-1}{2\Omega} - \sqrt{2gN}v \end{aligned} \tag{18}$$

provides an upper bound for all v and positive ω and Ω. We are free to rescale the variational parameters v, ω and Ω as we like, and we have considered the class of rescalings of the form $v \to N^a v$, $\omega \to N^b\omega$ and $\Omega \to N^c\Omega$. There are two different rescalings with non-trivial behavior in the large-N limit.

If we rescale the variational parameters as $v \to N^{1/2}v$, $\omega \to N\omega$, leaving Ω unchanged, we find after some algebra that the ground-state energy in this large-N limit is given by minimizing

$$E_{\text{var}} = N\left[\frac{\omega}{4} + \frac{\Omega}{4} - m^2v^2 + 4gv^4 + \frac{8gv^2}{\Omega} - \sqrt{2g}v\right]. \tag{19}$$

This in turn reduces to minimizing

$$E_{\text{var}} = N[-m^2v^2 + 4gv^4 + 2\sqrt{2gv^2} - \sqrt{2g}v] \tag{20}$$

with respect to v. Unsurprisingly, this is equivalent to minimizing our previous expression for the effective potential in the conventional large-N limit and is valid in region II.

A different scaling behavior, which we will show is valid in region I in the following section, is obtained if we perform rescalings $v \to N^{-1/6}v$, $\omega \to N^{1/3}\omega$ and $\Omega \to N^{-2/3}\Omega$. This yields a large-$N$ limit of the form

$$E_{\text{var}} = N^{1/3}\left[\frac{\omega}{4} + \frac{\Omega}{4} + \left(v^2 + \frac{1}{2\omega}\right)\frac{8g}{\Omega} - \sqrt{2g}v\right]. \tag{21}$$

Minimization of E_{var} with respect to the three parameters ω, Ω and v leads to the solution

$$\Omega = \left(\frac{16}{3}\right)^{2/3} g^{1/3} \tag{22}$$

$$\omega = 4\left(\frac{16}{3}\right)^{-1/3} g^{1/3} \tag{23}$$

$$v = 2^{-5/6} 3^{-2/3} g^{-1/6} \tag{24}$$

$$E_{\mathrm{var}} = \left(\frac{3}{2}\right)^{4/3} N^{1/3} g^{1/3}. \tag{25}$$

Numerically, this gives an upper bound on E_0 of approximately $1.71707 N^{1/3} g^{1/3}$. As we show below, this variational result is exact in the large-N limit in region I.

5. The Born–Oppenheimer approximation and the large-*N* limit

We now apply the two different scaling behaviors we have found in the previous section to the Hamiltonian directly. If the σ field is rescaled by $\sigma \to N^{1/2}\sigma$, the rescaled Hamiltonian is

$$H = -\frac{1}{2N}\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2}\frac{\partial^2}{\partial\vec{\pi}^2} - Nm^2\sigma^2 + 4gN\sigma^4 + 16g\sigma^2\vec{\pi}^2 - N\sqrt{2g}\sigma \tag{26}$$

which will turn out to be valid in region II. All of the terms in the Hamiltonian are of order *N*, except for the kinetic energy term for σ, which is of order $1/N$. This suggests the use of the Born–Oppenheimer approximation, in which heavy degrees of freedom are treated classically. In this approximation, the wavefunction is written as $\psi_{\mathrm{II}}(\vec{\pi},\sigma) = u_{\mathrm{II}}(\vec{\pi},\sigma)w_{\mathrm{II}}(\sigma)$ where $u_{\mathrm{II}}(\vec{\pi},\sigma)$ satisfies

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\vec{\pi}^2} + 16g\sigma^2\vec{\pi}^2\right]u_{\mathrm{II}}(\vec{\pi},\sigma) = \epsilon_{\mathrm{II}}(\sigma)u_{\mathrm{II}}(\vec{\pi},\sigma) \tag{27}$$

describing the $N-1$ $\vec{\pi}$ fields as harmonic oscillators with frequencies determined by σ. The total energy to leading order in N is again the effective potential V_{eff}. As we have seen, in region II there is a non-trivial solution for which the rescaled ground-state energy E_0/N has a finite, negative value as $N \to \infty$. In region I, we find that $E_0/N \to 0$ as $N \to \infty$; the vacuum expectation value of the original, un-rescaled σ also obeys $N^{-1/2}\langle\sigma\rangle \to 0$ in this limit as well.

The other rescaling is $\sigma \to N^{-1/6}\sigma$ combined with $\vec{\pi} \to N^{5/6}\vec{\pi}$. The rescaled Hamiltonian is

$$H = N^{1/3}\left[-\frac{1}{2}\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2N^2}\frac{\partial^2}{\partial\vec{\pi}^2} + 16g\sigma^2\vec{\pi}^2 - \sqrt{2g}\sigma - N^{-2/3}m^2\sigma^2 + 4gN^{-2}\sigma^4\right]. \tag{28}$$

With this rescaling, it appears energies will increase with N as $N^{1/3}$. We also see from the kinetic energy term for σ that the σ field retains its operator character as N goes to infinity, unlike the conventional large-N limit obtained in region II. We can immediately drop the σ^2 and σ^4 terms as irrelevant in the large-N limit. The relevance of the $\vec{\pi}$ kinetic field is less clear. If we again use hyperspherical coordinates for $\vec{\pi}$, we find that the effective Hamiltonian in the $l = 0$ sector can be written for large N as

$$H_{\mathrm{eff}} = N^{1/3}\left[-\frac{1}{2}\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2N^2}\frac{\partial^2}{\partial\rho^2} + \frac{1}{8\rho^2} + 16g\sigma^2\rho^2 - \sqrt{2g}\sigma\right], \tag{29}$$

where ρ is now the magnitude of the rescaled $\vec{\pi}$. It is clear that ρ has a mass of order N^2, and we again use the Born–Oppenheimer approximation, this time applied to ρ. The wavefunction is written as $\psi_{\mathrm{I}}(\vec{\pi}, \sigma) = u_{\mathrm{I}}(\sigma, \rho) w_{\mathrm{I}}(\rho)$ where $u_{\mathrm{I}}(\sigma, \rho)$ satisfies

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\sigma^2} + 16g\sigma^2\rho^2 - \sqrt{2g}\sigma\right] u_{\mathrm{I}}(\sigma, \rho) = \epsilon_{\mathrm{I}}(\rho) u_{\mathrm{I}}(\sigma, \rho), \tag{30}$$

which describes a particle in a harmonic potential. The ground-state energy is $2\sqrt{2g}\rho - 1/(32\rho^2)$. The energy of the combined system is given by minimizing

$$\epsilon_{\mathrm{I}}(\rho) + \frac{1}{8\rho^2} = \frac{3}{32\rho^2} + 2\sqrt{2g}\rho \tag{31}$$

which gives the ground-state energy in the large-N limit in region I to be exactly given by

$$E_0^{(\mathrm{I})} = \left(\frac{3}{2}\right)^{4/3} N^{1/3} g^{1/3} \tag{32}$$

which is identical to the result of the variational treatment in section 4.

We now have results for two different large-N limits. In one case, the ground-state energy is proportional to N

$$E_0^{(\mathrm{II})} = N \cdot \min_{\sigma} \left[-m^2\sigma^2 + 4g\sigma^4 + \tfrac{1}{2}\sqrt{32g\sigma^2} - \sqrt{2g}\sigma\right]. \tag{33}$$

In the other case, the ground-state energy

$$E_0^{(\mathrm{I})} = \left(\frac{3}{2}\right)^{4/3} N^{1/3} g^{1/3} \tag{34}$$

is proportional to $N^{1/3}$ and is always positive. If $E_0^{(\mathrm{II})}$ is negative, as it is in region II, it will be favored over $E_0^{(\mathrm{I})}$. In region I, the formula for $E_0^{(\mathrm{II})}$ appears to predict that the ground-state energy is zero. However, this is misleading: it is actually predicting that the ground-state energy is not growing linearly with N, but at some less rapid rate. This kind of behavior is shown by $E_0^{(\mathrm{I})}$, and the two expressions are in fact consistent. If we examine the behavior of the scaled ground-state energy E_0/N in the large-N limit, it is zero in region I and negative in region II. However, the true behavior of E_0 in region I is given by $E_0^{(\mathrm{I})}$. Note that similar considerations apply to the expectation value $\langle\sigma\rangle$. In region I, $\langle\sigma\rangle$ is decreasing as $N^{-1/6}$, while in region II it is increasing as $N^{1/2}$.

In the large-N limit, these two behaviors are completely incommensurate, and give rise to a first-order transition at $m^2/g^{2/3} = 3 \times 2^{1/3}$, the boundary between regions I and II. However, there can be no phase transition in quantum mechanics for any finite number of degrees of freedom. We can understand the behavior of these two solutions for finite but large values of N from the effective potential V_{eff} in figure 1. In the region of parameter space where $m^2/g^{2/3}$ is close to $3 \times 2^{1/3}$, V_{eff} has two distinct local minima, and there will be tunneling between these two minima, leading to a small splitting of the ground-state energy from that of the first excited state. This is very similar to the behavior of the double well with a small term linear in the coordinate x added. In this case, however, as N becomes large, the tunneling between the two minima is suppressed, leading to a first-order transition in the large-N limit.

6. The case $m^2 = 0$

In order to check our large-N result in region I, we study numerically the scaling behavior of the ground-state energy for the case $m^2 = 0$ for large values of N. This value for m^2 is in region I, where the ground-state energy scales as $N^{1/3}$ in the large-N limit. As we have seen in

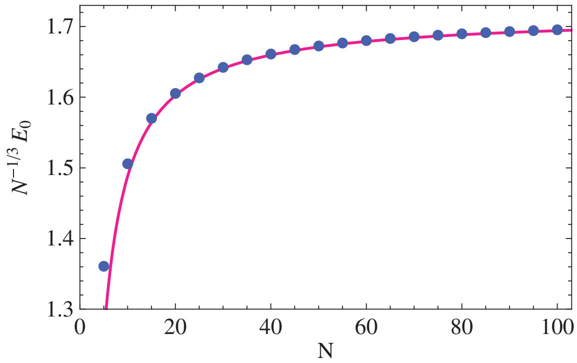

Figure 2. The variational estimate of the scaled ground-state energy versus N at $m^2 = 0$. The continuous line is a fit using the points from $N = 50$ to $N = 100$.

the previous section, the m^2 dependence disappears in this limit in region I, so the case $m^2 = 0$ gives the large-N behavior throughout this region. After rescaling the fields by $\sigma \to g^{-1/6}\sigma$ and $\vec{\pi} \to g^{-1/6}\vec{\pi}$, the radial Hamiltonian H_{rad} is given by

$$(gN)^{-1/3}H_{\text{rad}} = -\frac{1}{2}\frac{\partial^2}{\partial\sigma^2} + \frac{1}{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial\rho^2} + \frac{(N+2l-3)(N+2l-1)}{8\rho^2} + 16\sigma^2\rho^2\right) - \sqrt{2}\sigma - N^{-2/3}m^2g^{-2/3}\sigma^2 + 4N^{-2}\sigma^4. \tag{35}$$

Clearly, g can be set to 1 at $m^2 = 0$ with no loss of generality. Different angular momentum sectors do not mix, and we restrict ourselves to the $l = 0$ sector.

We have performed a series of variational calculations using a harmonic oscillator basis for both σ and ρ. In these calculations, a finite-dimensional square matrix is formed from the matrix elements of H_{rad} using a suitable basis. The lowest eigenvalue of this matrix is an upper bound on the ground-state energy. For ρ, these basis elements were chosen to be solutions of the reduced Hamiltonian for the $(N-1)$-dimensional harmonic oscillator. With the angular frequency scaled to 1, the general form of a basis function for ρ is $\rho^{N/2-1+l}L_{\frac{m-l}{2}}^{(N-3)/2+l}(\rho^2)\exp\left[-\frac{1}{2}\rho^2\right]$, where m is a non-negative integer, and has energy $m + (N-1)/2$. Basis sets of size n^2 were constructed using n basis elements for both σ and ρ. The optimum values for the angular frequencies associated with σ and ρ were determined by minimizing the lowest eigenvalue using a nine-element basis. These angular frequencies were then used with a 36-element basis to estimate the ground-state energy. In figure 2, we plot the 36-element estimate for the ground-state energy, scaled by a factor of $N^{-1/3}$, versus N. We also show a fit of the form

$$N^{-1/3}E_0 = a + \frac{b}{N}. \tag{36}$$

Although the fit was obtained using the points from $N = 50$ to $N = 100$, the agreement with the calculated energies is quite reasonable even at $N = 5$. The constants obtained from this fit are $a = 1.71705(4)$ and $b = -2.29(2)$, where the errors are estimated from Richardson extrapolation and varying the fitting conditions. These estimated errors do not represent the true systematic errors due, e.g., to truncation to a finite basis. Note the excellent agreement of a with the exact result $\left(\frac{3}{2}\right)^{4/3} \approx 1.71707$.

7. Conclusions

The PT-symmetric $O(N)$ model has been analyzed in a fairly complete way using its dual Hermitian form. The Hermitian form of the Hamiltonian has classical trajectories of finite energy that escape to infinity, and also $N - 1$ variables which are massless in perturbation theory. Nevertheless, the model can be proven to have only discrete energy eigenstates. There are two distinct large-N limits, with incommensurate scaling behavior. For $m^2/g^{2/3} > 3\times2^{1/3}$, the large-$N$ limit is similar in behavior to that of the Hermitian $O(N)$ model, with a ground-state energy depending on m and proportional to N. For $m^2/g^{2/3} < 3 \times 2^{1/3}$, the ground-state energy is proportional to $N^{1/3}$, and is independent of m throughout this region. The two regions are separated by a first-order phase transition in the limit $N \to \infty$. All of these results were obtained from the dual Hermitian form of the original PT-symmetric model. Although it was shown in [5] that the large-N limit of region II could be obtained from the PT-symmetric form, these arguments do not appear to give the detailed information for region I that can be obtained from the Hermitian form. It would be very desirable to achieve an understanding of this model for both regions using only the PT-symmetric form of the model. Such an understanding might be an important step in constructing PT-symmetric scalar field theories.

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